THE RIBAUCOUR TRANSFORMATION IN LIE SPHERE GEOMETRY

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ABSTRACT. We discuss the Ribaucour transformation of Legendre (contact) maps in its natural context: Lie sphere geometry. We give a simple conceptual proof of Bianchi's original Permutability Theorem and its generalisation by Dajczer–Tojeiro as well as a higher dimensional version with the combinatorics of a cube. We also show how these theorems descend to the corresponding results for submanifolds in space forms.

1. INTRODUCTION

A persistent and characteristic feature of integrable submanifold geometries is the existence of transformations of solutions. Examples include the Bäcklund transformations of surfaces of constant Gauss curvature and their generalisations [1, 13, 26]; Darboux transformations of isothermic surfaces [2, 9, 14, 25]; Eisenhart transformations of Guichard surfaces [19, §92]; Jonas transformations of R-congruences[1], to name but a few.

In all these cases, the transformation constructs new submanifolds of the desired kind from a known one with the help of a solution of an auxiliary completely integrable first order system of PDE. Moreover, some version of the *Bianchi Permutability Theorem* holds:

> *Given two transforms of a submanifold there is a fourth submanifold*
> *that is a simultaneous transform of the first two.*

This fourth submanifold is often unique and algebraically determined by the first three. We say that four submanifolds in this configuration form a *Bianchi quadrilateral*.

By the 1920's, it was realised that all these transformations had a common geometric foundation: they were all either *Ribaucour transformations* or *W-transformations* which latter are the projective-geometric analogue of the former under Lie's line-sphere correspondence (cf. [19]). The Ribaucour transformation was investigated in classical times for surfaces [3, 4, 19], and for triply and n-ply orthogonal systems [15, 3]. Moreover, the integrable systems approach to orthogonal systems and to discrete orthogonal (circular) nets has led to renewed interest in the Ribaucour transformation in modern times [5, 18, 17, 20, 22, 23, 27].

It is therefore the purpose of this paper to give a modern treatment of Ribaucour transformations and their permutability. Let us begin by describing what they are.

[1]In fact, Jonas transformations are Darboux transformations of isothermic surfaces in $\mathbb{R}^{2,2}$ [11].

Contemplate a pair of immersions $\mathfrak{f}, \hat{\mathfrak{f}} : \Sigma^k \to N$ of a k-manifold into a space-form. We say that $\hat{\mathfrak{f}}$ is a *Ribaucour transform* of $\mathfrak{f}$ if

(i) for each $p \in \Sigma^k$, there is a k-sphere $S(p)$ having first order contact with both $\mathfrak{f}$ and $\hat{\mathfrak{f}}$ at p.

(ii) the shape operators of $\mathfrak{f}$ and $\hat{\mathfrak{f}}$ commute.

Thus, in classical language, $\mathfrak{f}$ and $\hat{\mathfrak{f}}$ parametrise two submanifolds of a space form which envelop a congruence of spheres in such a way that curvature directions at corresponding points coincide.

We emphasise that these conditions are relatively mild: any submanifold is enveloped by sphere congruences (in Euclidean space, the congruence of (one-point compactified) tangent spaces is an example) and, at least in codimension one, any sphere congruence generically envelops two (possibly complex) submanifolds. Moreover, the condition on curvature directions is not difficult to arrange: any parallel submanifold to a given submanifold is a Ribaucour transform (in higher codimension, this example requires that the normal vector field joining the submanifolds be covariant constant). Thus the geometry of Ribaucour transforms is much less rigid than the integrable specialisations mentioned above. This is reflected in the following more general version of the permutability theorem which was proved by Bianchi [3, §354]:

Bianchi Permutability Theorem. *Given two Ribaucour transforms of a surface there are, generically, two 1-parameter families,* Demoulin families*, of surfaces, one containing the original surface and the other containing its two Ribaucour transforms, so that any member of one family is a Ribaucour transformation of any member of the other.*

Moreover, corresponding points on any of the surfaces in either family are concircular.

This result has recently been generalised to submanifolds in space-forms of arbitrary signature with arbitrary dimension and co-dimension by Dajczer–Tojeiro [12, 13].

Moreover, in the context of triply orthogonal systems, a higher dimensional version of the Bianchi Permutability Theorem has been obtained by Ganzha–Tsarev [20]:

Cube Theorem. *Given three initial Ribaucour transforms of a triply orthogonal system, a generic choice of three simultaneous Ribaucour transforms of two of them leads to an eighth orthogonal system which is a simultaneous Ribaucour transform of the latter three thus yielding the combinatorics of a cube — a "Bianchi cube".*

This higher dimensional version then generalises to any number of initial Ribaucour transforms and, in this way, gives rise to discrete orthogonal nets of any dimension by repeatedly applying the permutability theorem. In fact, this type of permutability theorem is central to integrable discretizations of smooth (geometric) integrable systems: it amounts to the "consistency condition" at the heart of the beautiful theory of Bobenko–Suris [7, 8]. Conversely, this theorem for discrete nets can be used to prove the permutability theorem for orthogonal systems of holonomic submanifolds by taking an appropriate limit [6].

The aim of this paper is to give a transparent and almost elementary proof of these permutability theorems for submanifolds in the realm of Lie sphere geometry. This is the correct context for a discussion of the Ribaucour transformation since the main ingredients of the theory, sphere congruences and the curvature directions of

enveloping submanifolds, are both Lie invariant notions. Our results descend to the respective permutability theorems for submanifolds in Riemannian space-forms (when one imposes the obvious regularity assumptions) and the Lie invariance of the constructions becomes manifest.

Here is how we will proceed: in Section 2 we swiftly rehearse the basics of Lie sphere geometry. Thus a sphere congruence is viewed as a map into a certain quadric and contact lifts of submanifolds as Legendre maps into the space of lines in that quadric. The enveloping relation is now one of incidence while the Ribaucour condition amounts to flatness of a certain normal bundle. In Section 3, we first prove the Bianchi Permutability Theorem for Legendre maps which we see amounts to the assertion that a certain rank four bundle is flat. Then we prove the analogue of the Cube Theorem of Ganzha–Tsarev for Legendre maps. At this point, our Lie sphere geometric discussion of the Permutability Theorems is complete: it only remains to show how our results imply those for submanifolds of space-forms. To this end, we briefly discuss Ribaucour transforms of submanifolds of a sphere in Section 4 and then, in Section 5, show how to construct Legendre maps (from the unit normal bundles of our submanifolds) and so find ourselves in the context of our main results. Here we make the effort to work in arbitrary codimension as applications such as those of [12] to submanifolds of constant sectional curvature require this. Finally, we conclude with a short discussion of an example in Section 6.

Remark. For clarity of exposition, we have limited ourselves to the case of definite signature but, modulo the imposition of additional assumptions of a generic nature, our entire analysis goes through in arbitrary signature. In particular, when applied to the Klein quadric (the projective light cone of $\mathbb{R}^{3,3}$), whose space of lines is the space of contact elements of $\mathbb{RP}^3$, we recover Bianchi's Permutability Theorem for focal surfaces of W-congruences.

2. Ribaucour sphere congruences

Contemplate $\mathbb{R}^{m+2,2}$: an $(m + 4)$-dimensional vector space with metric $(\,,\,)$ of signature $(m + 2, 2)$. Let $\mathcal{Q}$ denote the projective light-cone of $\mathbb{R}^{m+2,2}$:

$$\mathcal{Q} = \mathbb{P}(\mathcal{L}) = \{\mathbb{R}v \subset \mathbb{R}^{m+2,2} : (v, v) = 0, v \neq 0\}.$$

Thus $\mathcal{Q}$ is a manifold of dimension $m+2$ with a homogeneous action of $\mathrm{O}(m+2, 2)$.

Further, let $\mathcal{Z}$ denote the set of lines in $\mathcal{Q}$ or, equivalently, the Grassmannian of null 2-planes in $\mathbb{R}^{m+2,2}$. Then $\mathcal{Z}$ is an $\mathrm{O}(m + 2, 2)$-homogeneous contact manifold of dimension $2m + 1$.

The viewpoint of Lie sphere geometry is that $\mathcal{Q}$ parametrises (non-canonically) the set of oriented hyperspheres in S^{m+1} including those of zero radius. In this picture, $v, w \in \mathcal{L} \setminus \{0\}$ are orthogonal if and only if the corresponding hyperspheres are in oriented contact and so $\mathcal{Z}$ parametrises the space of contact elements on S^{m+1} (this is the origin of the contact structure of $\mathcal{Z}$).

Accordingly, we are lead to the following definitions:

Definition. A *sphere congruence* is a map $s : M^m \to \mathcal{Q}$ of an m-manifold.

Definition. A map $f : M^m \to \mathcal{Z}$ of an m-manifold is a *Legendre map* if, for all $\sigma_0, \sigma_1 \in \Gamma f$,

$$(\mathrm{d}\sigma_0, \sigma_1) \equiv 0.$$

Here, and below, we identify a map f of M^m into a (subspace of a) Grassmannian with the corresponding subbundle, also called f of $M^m \times \mathbb{R}^{m+2,2}$. The space of sections Γf therefore consists of maps $\sigma : M^m \to \mathbb{R}^{m+2,2}$ with each $\sigma(p) \in f(p)$, for $p \in M^m$.

The Legendre condition asserts that $df(TM^m)$ lies in the contact distribution along f and has the interpretation that f is the contact lift of a (tube around a) submanifold of S^{m+1} (see Section 5 below). This motivates our next definition:

Definition. A Legendre map $f : M^m \to \mathcal{Z}$ *envelops* a sphere congruence $s : M^m \to \mathcal{Q}$ if $s(p) \subset f(p)$, for all $p \in M^m$.

We are interested in the situation where two pointwise distinct Legendre maps $f, \hat{f} : M^m \to \mathcal{Z}$ envelop a common sphere congruence $s : M^m \to \mathcal{Q}$. Thus $s = f \cap \hat{f}$. Set

$$\mathcal{N}_{f,\hat{f}} = (f + \hat{f})/s.$$

Thus $\mathcal{N}_{f,\hat{f}}$ is a rank 2 subbundle of $s^\perp/s$. Now $s^\perp/s$ inherits a metric of signature $(m+1, 1)$ from that of $\mathbb{R}^{m+2,2}$ and this metric restricts to one of signature $(1, 1)$ on $\mathcal{N}_{f,\hat{f}}$ (otherwise each $f(p) + \hat{f}(p)$ would be a null 3-plane in $\mathbb{R}^{m+2,2}$). In particular, we have a well-defined orthogonal projection $\pi : s^\perp/s \to \mathcal{N}_{f,\hat{f}}$. Observe that, for $\sigma \in \Gamma s$, the Legendre condition on $f, \hat{f}$ gives

$$d\sigma \perp f + \hat{f}.$$

From this we see that, for $\nu \in \Gamma(f + \hat{f})$,

$$(d\nu, \sigma) = -(\nu, d\sigma) = 0$$

so that $d\nu$ takes values in $s^\perp$ while $\pi(d\sigma + s) = 0$. We therefore conclude:

Lemma 2.1. *There is a metric connection $\nabla^{f,\hat{f}}$ on $\mathcal{N}_{f,\hat{f}}$ given by*

$$\nabla^{f,\hat{f}}(\nu + s) = \pi(d\nu + s).$$

Remark. In case that $s : M^m \to \mathcal{Q}$ is an immersion, $\mathcal{N}_{f,\hat{f}}$ can be identified with the (weightless) normal bundle of s (with respect to the $O(m+2, 2)$-invariant conformal structure of signature $(m + 1, 1)$ on $\mathcal{Q}$) and we have an amusing identification of enveloping manifolds of s with null normal lines to s. In this setting, $\nabla^{f,\hat{f}}$ is the (conformally invariant) normal connection of the weightless normal bundle [10].

We are now in a position to make our basic definition:

Definition. Given Legendre maps $f, \hat{f} : M^m \to \mathcal{Z}$ enveloping a sphere congruence $s : M^m \to \mathcal{Q}$, that is, $s = f \cap \hat{f}$, we say that s is a *Ribaucour sphere congruence* if $\nabla^{f,\hat{f}}$ is flat.

In this case, $f, \hat{f}$ are said to be *Ribaucour transforms* of each other and that $(f, \hat{f})$ are a *Ribaucour pair*.

Remark. As we shall see in Section 5, this definition generically amounts to the classical notion of two k-dimensional submanifolds of S^{m+1} enveloping a congruence of k-spheres in such a way that curvature directions of corresponding normals coincide, cf. [4, 12, 13].

For future reference, we note:

Lemma 2.2. *For $\nu \in \Gamma f$, $\nabla^{f,\hat{f}}(\nu + s) = 0$ if and only if $d\nu \perp \hat{f}$.*

Proof. $\nabla^{f,\hat{f}}(\nu+s) = 0$ if and only if $\mathrm{d}\nu + s \perp (f + \hat{f}/s)$ or, equivalently, $\mathrm{d}\nu \perp f + \hat{f}$. But $\mathrm{d}\nu \perp f$ already since f is Legendre. $\qquad\square$

3. The Permutability Theorem

Let $\hat{f}_0, \hat{f}_1$ be Ribaucour transforms of a Legendre map $f_0 : M^m \to \mathcal{Z}$. The familiar assertion of Bianchi permutability is that there should be another Legendre map f_1 which is again a simultaneous Ribaucour transform of the $\hat{f}_i$. We say that
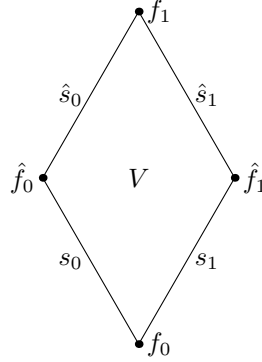

FIGURE 1. A Bianchi quadrilateral

the f_i's and the $\hat{f}_i$'s form a *Bianchi quadrilateral* the edges of which represent the enveloped Ribaucour sphere congruences, see Figure 1. In this section, we characterise the circumstances under which this assertion holds and show that, in this happy situation, a much stronger statement is available, cf. [3].

For this, we need some brief preliminaries. So let $(f_0, \hat{f}_0)$, $(f_0, \hat{f}_1)$ be distinct Ribaucour pairs with sphere congruences $s_i = f_0 \cap \hat{f}_i$. We impose the mild assumption that s_0 and s_1 are pointwise distinct so that $f_0 = s_0 \oplus s_1$. It follows that, for all $p \in M^m$, $\hat{f}_0 \cap \hat{f}_1 = \{0\}$ (otherwise $(\hat{f}_0(p) \cap \hat{f}_1(p)) \oplus f_0(p)$ is a null 3-plane) so that we may define a rank 4 subbundle V of $M^m \times \mathbb{R}^{m+2,2}$ by

$$V = \hat{f}_0 \oplus \hat{f}_1.$$

Since each $\hat{f}_i(p)$ is maximal isotropic in $\mathbb{R}^{m+2,2}$, V inherits a metric of signature $(2,2)$ from $\mathbb{R}^{m+2,2}$ and thus a metric connection ∇ by orthoprojection of d.

We may view any section $\sigma_0 \in \Gamma s_0$ as representing a section of $\mathcal{N}_{f_0,\hat{f}_1}$. We have:

Lemma 3.1. $\nabla^{f_0,\hat{f}_1}(\sigma_0 + s_1) = 0$ *if and only if* $\nabla\sigma_0 = 0$.

Proof. From Lemma 2.2, we know that $\nabla^{f_0,\hat{f}_1}(\sigma_0 + s_1) = 0$ if and only if $\mathrm{d}\sigma_0 \perp \hat{f}_1$. However, since $\hat{f}_0$ is Legendre, we already have $\mathrm{d}\sigma_0 \perp \hat{f}_0$ whence $\sigma_0 + s_1$ is parallel in $\mathcal{N}_{f_0,\hat{f}_1}$ if and only if $\mathrm{d}\sigma_0 \perp \hat{f}_0 \oplus \hat{f}_1 = V$, that is, $\nabla\sigma_0 = 0$. $\qquad\square$

By hypothesis, $\mathcal{N}_{f_0,\hat{f}_0}$, $\mathcal{N}_{f_0,\hat{f}_1}$ are both flat so that we have ∇-parallel sections $\sigma_i \in \Gamma s_i$, for $i = 0, 1$. From this we readily conclude:

Theorem 3.2. *Let f_1 be a simultaneous Ribaucour transform of $\hat{f}_0$ and $\hat{f}_1$ which is pointwise distinct from f_0. Then ∇ is flat and all four subbundles f_i, $\hat{f}_i$, $i = 0, 1$, are ∇-parallel.*

Proof. Applying the above analysis to f_1 in place of f_0 provides us with ∇-parallel sections $\hat{\sigma}_i \in \Gamma \hat{f}_i$ such that $f_1 = \langle \hat{\sigma}_0, \hat{\sigma}_1 \rangle$. These together with σ_0, σ_1 form a ∇-parallel frame for V and the result follows. □

Thus, if Bianchi permutability holds, V is flat. Locally, a converse, indeed, much stronger statement is available: assume that V is flat and M^m is simply-connected. Denote by $\mathcal{V}$ the vector space of ∇-parallel sections of V: evaluation at any fixed $p \in M^m$ gives us an isomorphism $\mathcal{V} \cong \mathbb{R}^{2,2}$. The projective light cone of $\mathcal{V}$ is a $(1,1)$-quadric and so ruled by two families of (real) lines (the α-lines and β-lines of twistor theory). Each family is parametrised by an $\mathbb{R}P^1$; lines of the same family do not intersect while each pair of lines from different families intersects in a unique point (thus our quadric is an isomorph of $\mathbb{R}P^1 \times \mathbb{R}P^1$).

A line in this quadric is the same as a map $f : M^m \to \mathcal{Z}$ with $f \subset V$ a ∇-parallel subbundle.

Lemma 3.3. *Let $f : M^m \to \mathcal{Z}$ be a map with $f \subset V$, where V is a flat $(2,2)$-bundle. Then f is ∇-parallel if and only if f is Legendre.*

Proof. Let $\sigma \in \Gamma f$. Then $d\sigma \perp f$ if and only if $d\sigma$ takes values in $f \oplus V^\perp$ or, equivalently, $\nabla \sigma$ takes values in f. □

We therefore have:

Theorem 3.4. *If V is a flat $(2,2)$-bundle, then there are two families $f_\alpha, \hat{f}_\beta : M^m \to \mathcal{Z}$, $\alpha, \beta \in \mathbb{R}P^1$, of Legendre maps such that:*

(i) $f_0, f_1 \in \{f_\alpha\}$;
(ii) $\hat{f}_0, \hat{f}_1 \in \{\hat{f}_\beta\}$;
(iii) $(f_\alpha, \hat{f}_\beta)$ *is a Ribaucour pair for all $\alpha, \beta \in \mathbb{R}P^1$.*

Proof. Only the third assertion requires any explanation: for this, note that $\mathcal{N}_{f_\alpha, \hat{f}_\beta}$ is spanned by sections which are represented by ∇-parallel sections of f_α and $\hat{f}_\beta$. By Lemma 3.1, these latter sections are $\nabla^{f_\alpha, \hat{f}_\beta}$-parallel so that $\mathcal{N}_{f_\alpha, \hat{f}_\beta}$ is flat whence $(f_\alpha, \hat{f}_\beta)$ is a Ribaucour pair. □

We call $\{f_\alpha\}$, $\{\hat{f}_\beta\}$ the *Demoulin families* of Legendre maps after their discoverer [15].

It remains to see when this beautiful state of affairs actually occurs: that is, starting from two Ribaucour transforms $\hat{f}_i$, $i = 0, 1$, of a Legendre map f_0, when is $V = \hat{f}_0 \oplus \hat{f}_1$ flat? For this, note that ∇ is metric and the sections σ_0, σ_1 are already ∇-parallel so that there is at most one direction in $\mathfrak{o}(2,2)$ for R^∇ to take values. Specifically:

Proposition 3.5. *Let $\hat{\sigma}_i \in \Gamma \hat{f}_i$ represent spanning sections of $\hat{f}_i/s_i$, $i = 0, 1$. Then V is flat if and only if $(R^\nabla \hat{\sigma}_0, \hat{\sigma}_1) \equiv 0$.*

In particular, we can weaken the hypotheses of Theorem 3.2:

Proposition 3.6. *V is flat if and only if f admits a Legendre complement $f_1 : M^m \to \mathcal{Z}$ in V: $V = f \oplus f_1$*

Proof. We can choose sections $\hat{\sigma}_i$ of $f_1 \cap \hat{f}_i$ dual to σ_i: $(\hat{\sigma}_0, \sigma_1) = (\hat{\sigma}_1, \sigma_0) \equiv 1$. Then $\mathrm{d}\hat{\sigma}_0 \perp \sigma_0, \hat{\sigma}_0$ since $\hat{f}_0$ is Legendre, while

$$(\mathrm{d}\hat{\sigma}_0, \sigma_1) = -(\hat{\sigma}_0, \mathrm{d}\sigma_1) = 0$$

since σ_1 is ∇-parallel. Finally $\mathrm{d}\hat{\sigma}_0 \perp \hat{\sigma}_1$ as f_1 is Legendre. Therefore $\hat{\sigma}_0$ is ∇-parallel. Similarly, $\hat{\sigma}_1$ is parallel so that V is flat since it is spanned by parallel sections. □

In Section 5, we shall see that, generically, the bundle V defined by two Ribaucour transforms of a Legendre map is automatically flat so that the Permutability Theorem described in Theorem 3.4 does indeed hold. However, we shall show by an example that V can fail to be flat and so the Permutability Theorem fails also.

We now turn to a higher dimensional of the Permutability Theorem analogous to that obtained in [20] for orthogonal systems. For this, start with a Legendre map f and three Ribaucour transforms thereof $\hat{f}_0, \hat{f}_1, \hat{f}_2$. Assume that the Bianchi Permutability Theorem applies so that we have three more Legendre maps f_0, f_1, f_2 forming three Bianchi quadrilaterals with vertices $f, \hat{f}_i, f_j, \hat{f}_k$ (i, j, k distinct). One can now attempt to apply the theorem again starting with each $\hat{f}_i$ and its Ribaucour transforms f_j, f_k. The astonishing fact is that there is a single Legendre map $\hat{f}$ which is a simultaneous Ribaucour transform of all the f_i so that we obtain a configuration of eight Legendre maps forming six Bianchi quadrilaterals with the combinatorics of a cube, a *Bianchi cube*, whose vertices are Legendre maps and whose edges are the enveloped Ribaucour sphere congruences. The situation is illustrated in Figure 2 where the eighth surface $\hat{f}$ has been placed at infinity.
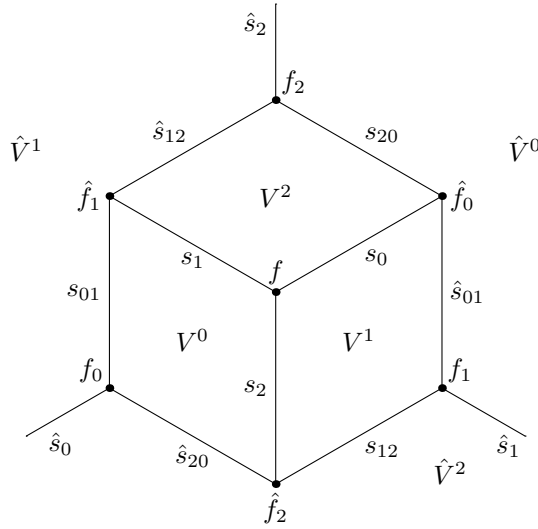

FIGURE 2. A Bianchi cube

This result needs some mild genericity hypotheses. Here is the precise statement:

Theorem 3.7. *Let $f, \hat{f}_0, \hat{f}_1, \hat{f}_2, f_0, f_1, f_2$ be Legendre maps with each $f, \hat{f}_i, f_j, \hat{f}_k$ a Bianchi quadrilateral for i, j, k distinct.*

Assume that, for i, j, k distinct,

$$(3.1a) \qquad\qquad\qquad \hat{f}_i \not\subset \hat{f}_j \oplus \hat{f}_k$$

$$(3.1b) \qquad\qquad\qquad f_i \cap f_j = \{0\}$$

$$(3.1c) \qquad\qquad\qquad f_i \not\subset f_j \oplus f_k.$$

Then there is a unique Legendre map $\hat{f}$ which is a simultaneous Ribaucour transform of f_0, f_1, f_2.

Remarks.

1. The hypothesis (3.1a) fails exactly when all the $\hat{f}_i$ lie in a single Demoulin family. In this degenerate case, any other $\hat{f}$ in the same family satisfies the conclusion of the theorem (without the uniqueness assertion!).
2. The hypothesis (3.1b) is satisfied for generic choices of the f_i in their respective Demoulin families. Indeed, for given f_0, the set of f_1, f_2 satisfying (3.1b) with f_0 is non-empty[2] and (Zariski) open.
3. With a little effort, one can show that hypothesis (3.1c) follows from (3.1a).

We begin the proof of Theorem 3.7 by setting up notation: given the seven Legendre maps of the statement of the theorem, define bundles of $(2, 2)$-planes by

$$V^0 := \hat{f}_1 \oplus \hat{f}_2 = f \oplus f_0, \quad V^1 := \hat{f}_2 \oplus \hat{f}_0 = f \oplus f_1, \quad V^2 := \hat{f}_0 \oplus \hat{f}_1 = f \oplus f_2.$$

By (3.1a), $\hat{f}_i \not\subset V^i$ from which it follows that

$$V := \hat{f}_0 + \hat{f}_1 + \hat{f}_2$$

is a bundle of $(3, 2)$-planes[3]. Further, by (3.1b), we have three more bundles of $(2, 2)$-planes:

$$\hat{V}^2 := f_0 \oplus f_1, \qquad\qquad \hat{V}^0 := f_1 \oplus f_2, \qquad\qquad \hat{V}^1 := f_2 \oplus f_0.$$

We label the sphere congruences implementing the various Ribaucour transformations as in Figure 2.

Now for the uniqueness assertion: if $\hat{f}$ is a simultaneous Ribaucour transformation of the f_i, set $\hat{s}_i = \hat{f} \cap f_i \subset \hat{V}^i \cap f_i$. By (3.1c), $f_i \not\subset \hat{V}^i$ whence $\hat{s}_i = \hat{V}^i \cap f_i$ and so is determined by the first seven Legendre maps. Thus $\hat{f}$ is so determined also.

To see that a simultaneous Ribaucour transform $\hat{f}$ exists, start by defining null line-bundles $\hat{s}_i = \hat{V}^i \cap f_i$. One readily checks that, for distinct i, j, k, $\hat{s}_i \neq \hat{s}_j$ and

$$\hat{s}_i \subset \hat{s}_j \oplus \hat{s}_k$$

so that we have a well-defined bundle of 2-planes

$$\hat{f} = \hat{s}_i \oplus \hat{s}_j$$

which is null since it contains three distinct null lines (the $\hat{s}_k$).

In view of Proposition 3.6, the only thing left to prove is that $\hat{f}$ is Legendre. Our proof of this hinges on the existence of sections of the various sphere congruences with the property that their sum around any vertex of the cube vanishes. *A priori*, this requirement seems overdetermined (even around a single face!) but we will be able to construct such sections from a consistent choice of normals to the faces. This is an entirely algebraic matter so we consider the situation at a single point.

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[2] For example, we could take $f_1 = f_2 = f$

[3] To see that V has non-degenerate metric note that any element of $V \cap V^\perp$ together with any f_i would span a null 3-plane.

Thus we contemplate a configuration of null 2-planes, null lines and $(2,2)$ planes in $\mathbb{R}^{3,2}$ assigned, respectively, to the vertices, edges and faces of a combinatorial cube with the property that each line, resp. $(2,2)$-plane, corresponding to an edge, resp. face, is given by the intersection, resp. span, of the null 2-planes corresponding to incident vertices. We label the components of this configuration as in Figure 2.

In line with the hypotheses of Theorem 3.7, we assume that the V^i are pairwise distinct, from which it follows that $V^i \neq \hat{V}^j$, for all i, j, and that the $\hat{V}^j$ are pairwise distinct also.

Now let $\nu_i, \hat{\nu}_j$ be unit normals in $\mathbb{R}^{3,2}$ to $V^i, \hat{V}^j$. The situation along an edge of our cube is given by:

Lemma 3.8. *Let $f, \hat{f}_2$ be null 2-planes and V^0, V^1 $(2,2)$-planes in $\mathbb{R}^{3,2}$ such that*

(i) $V^0 \neq V^1$;
(ii) $f + \hat{f}_2 \subset V^0 \cap V^1$;
(iii) $s_2 := f \cap \hat{f}_2$ *is a null line.*

Let ν_0, ν_1 be unit normal vectors to V^0, V^1. Then

$$\nu_1 = \varepsilon \nu_0 + \sigma,$$

where $\varepsilon = (\nu_0, \nu_1) = \pm 1$ and $\sigma_2 \in s_2 \setminus \{0\}$.

Proof. Write $\nu_1 = \varepsilon \nu_0 + \sigma_2$ with $\sigma_2 \in V^0$ and $\varepsilon = (\nu_0, \nu_1)$. Note that σ_2 is non-zero since the ν_i are not collinear. The ν_i are orthogonal to $f, \hat{f}_2$ whence σ_2 is also. Thus σ_2 lies in $V^0 \cap (f + \hat{f}_2)^{\perp} = s_2$ and we are done since $(\nu_1, \nu_1) = 1$ delivers $\varepsilon^2 = 1$. $\square$

Thus we have a function ε with values in $\mathbb{Z}_2$ defined on the edges of the cube given by the inner product of adjacent normals. Concerning this, we have:

Lemma 3.9. *Let s_a, s_b, s_c be edges meeting at a vertex. Then*

$$\varepsilon(s_a)\varepsilon(s_b)\varepsilon(s_c) = 1.$$

Proof. For definiteness, take the vertex to be f. Apply Lemma 3.8 to each edge to get

$$\nu_1 = \varepsilon_2 \nu_0 + \sigma_2 \qquad \nu_2 = \varepsilon_0 \nu_1 + \sigma_0 \qquad \nu_0 = \varepsilon_1 \nu_2 + \sigma_1$$

whence

$$(1 - \varepsilon_0 \varepsilon_1 \varepsilon_2)\nu_1 = \varepsilon_2 \varepsilon_1 \sigma_0 + \varepsilon_2 \sigma_1 + \sigma_2.$$

The σ_i all lie in f so that the right hand side of this is isotropic and the lemma follows. $\square$

Lemma 3.10. *There is a choice of normals for which $\varepsilon \equiv 1$, that is, for all $i \neq j$,*

$$(\nu_i, \nu_j) = (\nu_i, \hat{\nu}_j) = (\hat{\nu}_i, \hat{\nu}_j) = 1.$$

Proof. Apply Lemma 3.9 at the vertex f: either all ε_i are 1 or two of them, $\varepsilon_1, \varepsilon_2$ say, are -1 in which case replace ν_0 by $-\nu_0$ to get a choice with all $\varepsilon_i = 1$.

The same argument at $\hat{f}$ gives us $\hat{\nu}_i$ with all $\varepsilon = 1$ around $\hat{f}$. Moreover, changing the signs of all $\hat{\nu}_i$ at once, if necessary, we may assume that all $\varepsilon = 1$ around f_1. At each remaining vertex in turn, we have that two of the $\varepsilon = 1$ whence, by Lemma 3.9, the third is also. $\square$

Remark. This argument is cohomological: view ε as a 1-cochain on the octahedron dual to the cube. Then Lemma 3.9 says that ε is a cocycle while Lemma 3.10 says that it is a coboundary (of the 0-cochain that consistently orients our normals).

Proposition 3.11. *In the situation of theorem 3.7, there are non-zero sections* $\sigma_i, \hat{\sigma}_j, \sigma_{ij}, \hat{\sigma}_{jk}$ *of the participating sphere congruences the sum of which around any vertex is zero:*

$$\sigma_0 + \sigma_1 + \sigma_2 = 0 \quad \hat{\sigma}_0 + \hat{\sigma}_1 + \hat{\sigma}_2 = 0 \quad \sigma_{ki} + \sigma_i + \hat{\sigma}_{ij} = 0 \quad \hat{\sigma}_{ki} + \hat{\sigma}_i + \sigma_{ij} = 0,$$

for all (i, j, k) *a cyclic permutation of* $(1, 2, 3)$.

Proof. With normals chosen as in Lemma 3.10, use Lemma 3.8 to define sections by:

$$\begin{aligned}
\nu_0 &= \nu_2 + \sigma_1 = \hat{\nu}_2 + \hat{\sigma}_{20}, & \hat{\nu}_0 &= \hat{\nu}_2 + \hat{\sigma}_1 = \nu_2 + \sigma_{20}, \\
\nu_1 &= \nu_0 + \sigma_2 = \hat{\nu}_0 + \hat{\sigma}_{01}, & \hat{\nu}_1 &= \hat{\nu}_0 + \hat{\sigma}_2 = \nu_0 + \sigma_{01}, \\
\nu_2 &= \nu_1 + \sigma_0 = \hat{\nu}_1 + \hat{\sigma}_{12}, & \hat{\nu}_2 &= \hat{\nu}_1 + \hat{\sigma}_0 = \nu_1 + \sigma_{12}.
\end{aligned}$$

These clearly have the desired property. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

With this preparation in hand, we can now complete the proof of Theorem 3.7 by showing that $\hat{f}$ is Legendre. We compute:

$$(3.2\text{a}) \qquad\qquad (\mathrm{d}\hat{\sigma}_0, \hat{\sigma}_1) = (\mathrm{d}\sigma_{01} + \mathrm{d}\hat{\sigma}_{20}, \sigma_{12} + \hat{\sigma}_{01})$$

$$(3.2\text{b}) \qquad\qquad\qquad = (\mathrm{d}\sigma_{01}, \sigma_{12}) + (\mathrm{d}\sigma_{01}, \hat{\sigma}_{01}) + (\mathrm{d}\hat{\sigma}_{20}, \hat{\sigma}_{01})$$

where we have used Proposition 3.11 at f_0, f_1 for (3.2a) and that f_2 is Legendre for (3.2b). Now

$$(\mathrm{d}\sigma_{01}, \sigma_{12}) = -(\mathrm{d}\sigma_{01}, \sigma_2 + \hat{\sigma}_{20}) = -(\mathrm{d}\sigma_{01}, \sigma_2)$$

thanks to Proposition 3.11 at $\hat{f}_2$ and the fact that f_0 is Legendre. Similarly,

$$(\mathrm{d}\hat{\sigma}_{20}, \hat{\sigma}_{01}) = -(\mathrm{d}\sigma_2 + \mathrm{d}\sigma_{12}, \hat{\sigma}_{01}) = -(\mathrm{d}\sigma_2, \hat{\sigma}_{01}).$$

On the other hand, since f_2, f_0 are Legendre,

$$\begin{aligned}
0 = (\mathrm{d}\hat{\sigma}_{12}, \sigma_{20}) &= (\mathrm{d}\sigma_{01} + \mathrm{d}\sigma_1, \sigma_0 + \hat{\sigma}_{01}) \\
&= (\mathrm{d}\sigma_{01}, \hat{\sigma}_{01}) + (\mathrm{d}\sigma_1, \hat{\sigma}_{01}) + (\mathrm{d}\sigma_{01}, \sigma_0)
\end{aligned}$$

and substituting all this back into (3.2) gives

$$\begin{aligned}
(\mathrm{d}\hat{\sigma}_0, \hat{\sigma}_1) &= -(\mathrm{d}\sigma_{01}, \sigma_2) - (\mathrm{d}\sigma_{01}, \sigma_0) - (\mathrm{d}\sigma_1, \hat{\sigma}_{01}) - (\mathrm{d}\sigma_2, \hat{\sigma}_{01}) \\
&= (\mathrm{d}\sigma_{01}, \sigma_1) + (\mathrm{d}\sigma_0, \hat{\sigma}_{01}) = 0
\end{aligned}$$

because $\hat{f}_0, \hat{f}_1$ are Legendre.

Remark. Viewing Theorem 3.7 as a 3-dimensional version of the permutability theorem, it is natural to enquire as to whether higher dimensional versions of the result are available. This is indeed the case: firstly the 4-dimensional version is equivalent to the usual Bianchi permutability theorem for quadrilaterals of discrete Ribaucour transforms of 2-dimensional discrete orthogonal nets and this has been proved in the context of conformal geometry by the second author [21, §§8.5.8 and 8.5.9] using a rather intricate but elementary argument relying solely on Miguel's theorem. Thereafter, a simple induction argument using the uniqueness assertion of Theorem 3.7 establishes the result in any higher dimension.

4. Ribaucour transforms in Riemannian geometry

Let us now make contact with the more familiar Riemannian geometry of the unit sphere[4] $S^{m+1} \subset \mathbb{R}^{m+2}$.

An immersion $\mathfrak{f} : \Sigma^k \to S^{m+1}$ of a k-manifold envelops a congruence of k-spheres if, for each $q \in \Sigma^k$, there is a k-sphere $S(q) \subset S^{m+1}$ such that

$$\mathfrak{f}(q) \in S(q), \qquad \mathrm{d}\mathfrak{f}(T_q\Sigma^k) = T_{\mathfrak{f}(q)}S(q).$$

Thus $S(q)$ is the intersection of S^{m+1} with an affine $(k+1)$-plane $\mathfrak{f}(q) + W_0(q)$ with $\mathrm{d}\mathfrak{f}(T_q\Sigma^k) \subset W_0(q)$.

A second (pointwise distinct) immersion $\hat{\mathfrak{f}} : \Sigma^k \to S^{m+1}$ envelops the same sphere congruence exactly when $\mathfrak{f}(q) + W_0(q) = \hat{\mathfrak{f}}(q) + W_0(q)$ and $\mathrm{d}\hat{\mathfrak{f}}(T_q\Sigma^k) \subset W_0(q)$. Otherwise said:

Lemma 4.1. $\mathfrak{f}, \hat{\mathfrak{f}} : \Sigma^k \to S^{m+1}$ *envelop a common sphere congruence if and only if*

$$(4.1) \qquad \langle \mathrm{d}\mathfrak{f}(T\Sigma^k), \hat{\mathfrak{f}} - \mathfrak{f} \rangle = \langle \mathrm{d}\hat{\mathfrak{f}}(T\Sigma^k), \hat{\mathfrak{f}} - \mathfrak{f} \rangle.$$

In this situation, we may therefore define $r \in \Gamma\mathrm{End}(T\Sigma^k)$ and a 1-form α by $\mathrm{d}\hat{\mathfrak{f}} = \mathrm{d}\mathfrak{f} \circ r + \alpha(\hat{\mathfrak{f}} - \mathfrak{f})$ or, equivalently,

$$(4.2) \qquad \mathrm{d}\hat{\mathfrak{f}} - \alpha\hat{\mathfrak{f}} = \mathrm{d}\mathfrak{f} \circ r - \alpha\mathfrak{f}.$$

Taking the norm-squared of this last and subtracting $\alpha \otimes \alpha$ yields

$$(4.3) \qquad (\mathrm{d}\hat{\mathfrak{f}}, \mathrm{d}\hat{\mathfrak{f}}) = (\mathrm{d}\mathfrak{f} \circ r, \mathrm{d}\mathfrak{f} \circ r)$$

so that r intertwines the metrics on Σ^k induced by $\mathfrak{f}$ and $\hat{\mathfrak{f}}$. This tensor has a basic role to play in what follows.

Now denote by W_0 the rank $(k+1)$ subbundle of $\Sigma^k \times \mathbb{R}^{m+2}$ defined by (4.1). Moreover, let $\rho(q)$ be reflection in the hyperplane orthogonal to $\hat{\mathfrak{f}}(q) - \mathfrak{f}(q)$ so that $\rho : \Sigma^k \to O(m+2)$. Then $\rho\mathfrak{f} = \hat{\mathfrak{f}}$ and $\rho W_0 = W_0$ (since $\hat{\mathfrak{f}} - \mathfrak{f} \in \Gamma W_0$). Now $\mathrm{d}\mathfrak{f}(T\Sigma^k) = W_0 \cap \langle f \rangle^\perp$ and similarly for $\hat{\mathfrak{f}}$ whence $\rho\mathrm{d}\mathfrak{f}(T\Sigma^k) = \mathrm{d}\hat{\mathfrak{f}}(T\Sigma^k)$ and we conclude that ρ provides a metric isomorphism[5] $\rho : N_\mathfrak{f} \to N_{\hat{\mathfrak{f}}}$ between the normal bundles of $\mathfrak{f}$ and $\hat{\mathfrak{f}}$. This will allow us to compare curvature directions of $\mathfrak{f}$ and $\hat{\mathfrak{f}}$.

For this, let $\xi \in \Gamma N_\mathfrak{f}$ and set $\lambda = (\xi, \hat{\mathfrak{f}})/((\mathfrak{f}, \hat{\mathfrak{f}}) - 1)$. Then $\xi - \lambda\mathfrak{f} \perp \hat{\mathfrak{f}} - \mathfrak{f}$ and so is fixed by ρ:

$$(4.4) \qquad \xi - \lambda\mathfrak{f} = \rho\xi - \lambda\hat{\mathfrak{f}}.$$

Together with (4.2), this yields

$$\big(\mathrm{d}(\rho\xi - \lambda\hat{\mathfrak{f}}), \mathrm{d}\hat{\mathfrak{f}} - \alpha\hat{\mathfrak{f}}\big) = \big(\mathrm{d}(\xi - \lambda\mathfrak{f}), \mathrm{d}\mathfrak{f} \circ r - \alpha\mathfrak{f}\big)$$

to both sides of which we add $\mathrm{d}\lambda \otimes \alpha$ to get

$$\big(\mathrm{d}(\rho\xi - \lambda\hat{\mathfrak{f}}), \mathrm{d}\hat{\mathfrak{f}}\big) = \big(\mathrm{d}(\xi - \lambda\mathfrak{f}), \mathrm{d}\mathfrak{f} \circ r\big).$$

We express this last in terms of the shape operators A, $\hat{A}$ of $\mathfrak{f}$ and $\hat{\mathfrak{f}}$:

$$\big(\mathrm{d}\hat{\mathfrak{f}}(\hat{A}^{\rho\xi} + \lambda), \mathrm{d}\hat{\mathfrak{f}}\big) = \big(\mathrm{d}\mathfrak{f}(A^\xi + \lambda), \mathrm{d}\mathfrak{f} \circ r\big)$$

---

[4]All our constructions have manifest conformal invariance so we could have chosen any other $(m+1)$-dimensional space form as our Riemannian context.

[5]With a little more effort, one can also show that ρ intertwines the normal connections on $N_\mathfrak{f}$ and $N_{\hat{\mathfrak{f}}}$ also.

and use (4.3) to conclude that r also intertwines shape operators:

$$(4.5) \qquad r \circ (\hat{A}^{\rho\xi} + \lambda) = A^\xi + \lambda.$$

Now $\hat{A}^{\rho\xi}$ is symmetric with respect to the metric induced by $\hat{\mathfrak{f}}$ so that (computing transposes with respect to the metric induced by $\mathfrak{f}$)

$$r^T r \hat{A}^{\rho\xi} = (\hat{A}^{\rho\xi})^T r^T r,$$

which, together with (4.5), yields

$$(4.6) \qquad r^T(A^\xi + \lambda) = (A^\xi + \lambda)r.$$

With this in hand, we can explain the significance of r for us:

Proposition 4.2. *If r is symmetric with respect to the metric induced by $\mathfrak{f}$ then corresponding shape operators of $\mathfrak{f}$ and $\hat{\mathfrak{f}}$ commute: $[A^\xi, \hat{A}^{\rho\xi}] = 0$, for all $\xi \in N_\mathfrak{f}$.*

Conversely, if, for some $\xi \in N_\mathfrak{f}$, $[A^\xi, \hat{A}^{\rho\xi}] = 0$ and, additionally, $A^\xi + \lambda$ is invertible[6], then r is symmetric.

Proof. From (4.5) we have

$$0 = [A^\xi + \lambda, r](\hat{A}^{\rho\xi} + \lambda) + r[A^\xi, \hat{A}^{\rho\xi}].$$

Now, if r is symmetric, (4.6) gives $[A^\xi + \lambda, r] = 0$ and the conclusion follows from the invertibility of r.

For the converse, given $\xi \in N_\mathfrak{f}$ with $[A^\xi, \hat{A}^{\rho\xi}] = 0$ and $A^\xi + \lambda$, equivalently $A^\xi + \lambda$, invertible, we first deduce that $[A^\xi + \lambda, r] = 0$ and then, from (4.6), that r is symmetric.   □

The notion of Ribaucour transform currently available in the literature [4, 12, 13, 19, 26] involves a pair of k-dimensional submanifolds enveloping a congruence of k-spheres so that curvature directions of corresponding normals coincide (that is, corresponding shape operators commute). In view of Proposition 4.2, we propose the following

Definition. Immersions $\mathfrak{f}, \hat{\mathfrak{f}} : \Sigma^k \to S^{m+1}$ enveloping a congruence of k-spheres are a *Ribaucour pair* if r is symmetric.

Remark. Given such a Ribaucour pair, every A^ξ commutes with r. Thus, in the generic case where r has distinct eigenvalues, all shape operators of $\mathfrak{f}$ must commute with each other so that the normal bundle of $\mathfrak{f}$ is flat (whence $\hat{\mathfrak{f}}$ has flat normal bundle also).

5. FROM RIEMANNIAN TO LIE SPHERE GEOMETRY

Given $\mathfrak{f}, \hat{\mathfrak{f}} : \Sigma^k \to S^{m+1}$ enveloping a congruence of k-spheres, we are going to construct Legendre maps $f, \hat{f} : M^m \to \mathcal{Z}$ enveloping $s : M^m \to \mathcal{Q}$ where M^m is the unit normal bundle of $\mathfrak{f}$.

We shall show:

(i) $\mathcal{N}_{f,\hat{f}}$ is flat if and only if r is symmetric so that our two notions of Ribaucour pair correspond.

(ii) Given Ribaucour transforms $\hat{\mathfrak{f}}_0, \hat{\mathfrak{f}}_1$ with corresponding r_0, r_1, the bundle $V = \hat{f}_0 \oplus \hat{f}_1$ of Section 3 is flat if and only if $[r_0, r_1] = 0$.

---

[6]When Σ^k is a hypersurface, this is precisely the condition that our sphere congruence contains no curvature spheres.

(iii) In this latter situation, all the Legendre maps $f_s, \hat{f}_t$ participating in the Permutability Theorem also arise from maps $\mathfrak{f}_s, \hat{\mathfrak{f}}_t : \Sigma^k \to S^{m+1}$.

For all this, fix orthogonal unit time-like vectors $t_0, t_1 \in \mathbb{R}^{m+2,2}$ and set $\mathbb{R}^{m+2} = \langle t_0, t_1 \rangle^{\perp}$. Thus

$$\mathbb{R}^{m+2,2} = \mathbb{R}^{m+2} \oplus \langle t_0 \rangle \oplus \langle t_1 \rangle$$

is an orthogonal decomposition. The quadric $\mathcal{Q}$ splits as a disjoint union $\mathcal{Q} = \mathcal{Q}_0 \cup \mathcal{Q}_+$ where

$$\mathcal{Q}_0 = \{ \langle v \rangle \in \mathcal{Q} : v \perp t_1 \}$$

is the space of *point spheres*. Note that $x \mapsto \langle x + t_0 \rangle : S^{m+1} \to \mathcal{Q}_0$ is a diffeomorphism. We therefore define $\phi, \hat{\phi} : \Sigma^k \to \mathcal{L} \subset \mathbb{R}^{m+2,2}$ by

$$\phi = \mathfrak{f} + t_0, \qquad \hat{\phi} = \hat{\mathfrak{f}} + t_0.$$

Now let M^m be the unit normal bundle of $\mathfrak{f}$ with bundle projection $\pi : M^m \to \Sigma^k$. For $\xi \in M^m$ with $\pi(\xi) = q$, define $\lambda(\xi)$ by

$$\lambda(\xi) = (\xi, \hat{\phi}(q))/(\phi(q), \hat{\phi}(q)) = (\xi, \hat{f}(q))/\big((\mathfrak{f}(q), \hat{\mathfrak{f}}(q)) - 1\big).$$

Observe that $\xi - \lambda(\xi)\phi(q) \perp \phi(q), \hat{\phi}(q)$ while (4.4) gives:

$$\xi - \lambda(\xi)\phi(q) = \rho\xi - \lambda(\xi)\hat{\phi}(q).$$

Here the left hand side is clearly orthogonal to $\mathrm{d}\phi(T_q\Sigma^k) = \mathrm{d}\mathfrak{f}(T_q\Sigma^k)$ while the right hand side is orthogonal to $\mathrm{d}\hat{\phi}(T_q\Sigma^k)$. Thus, defining $\sigma : M^m \to \mathcal{L} \subset \mathbb{R}^{m+2,2}$ by

$$\sigma(\xi) = \xi - \lambda(\xi)\phi(q) + t_1 = \rho\xi - \lambda(\xi)\hat{\phi}(q) + t_1,$$

we readily conclude:

(5.1) $$(\phi \circ \pi, \sigma) = (\hat{\phi} \circ \pi, \sigma) = 0$$

(5.2) $$(d(\phi \circ \pi), \sigma) = (d(\hat{\phi} \circ \pi), \sigma) = 0$$

We therefore have Legendre maps $f, \hat{f} : M^m \to \mathcal{Z}$ with $f \cap \hat{f} = s : M^m \to \mathcal{Q}$ as follows:

$$f = \langle \phi \circ \pi, \sigma \rangle, \qquad \hat{f} = \langle \hat{\phi} \circ \pi, \sigma \rangle, \qquad s = \langle \sigma \rangle.$$

Remark. Here is the geometry of the situation: a unit normal ξ to $\mathfrak{f}$ at q defines a contact element $\langle \xi \rangle^{\perp} \subset T_{\mathfrak{f}(q)} S^{m+1}$ containing $\mathrm{d}\mathfrak{f}(T_q\Sigma^k)$: this is $f(\xi)$. Among the hyperspheres sharing this contact element is exactly one which is also tangent to $\hat{\mathfrak{f}}$ at q: this is $s(\xi)$.

We have (at last!) found ourselves in the setting of Section 2 and so can investigate $\mathcal{N}_{f,\hat{f}}$. For this, first contemplate the bundle $\langle \phi, \hat{\phi} \rangle \to \Sigma^k$ with metric and connection $\nabla^{\phi,\hat{\phi}}$ inherited from $\Sigma^k \times \mathbb{R}^{m+2,2}$. Using $(\sigma, t_1) \equiv -1$ and the fact that $\phi \circ \pi + s$, $\hat{\phi} \circ \pi + s$ are spanning sections of $\mathcal{N}_{f,\hat{f}}$, it is not difficult to show that:

Lemma 5.1. *There is a metric, connection-preserving isomorphism between $\mathcal{N}_{f,\hat{f}}$ and the pull-back $\pi^{-1}\langle \phi, \hat{\phi} \rangle$ given by*

$$\tau + s \mapsto \tau + (\tau, t_1)\sigma.$$

In particular, since π is a submersion, we conclude:

Proposition 5.2. $\mathcal{N}_{f,\hat{f}}$ *is flat if and only if* $\langle \phi, \hat{\phi} \rangle$ *is flat*[7]

---

[7]This last is the definition of Ribaucour transform adopted by Burstall–Calderbank [10] in their conformally invariant treatment of this topic.

The latter condition is easy to characterise: define the second fundamental form $\beta \in \Omega^1_{\Sigma^k} \otimes \mathrm{Hom}(\langle \phi, \hat{\phi} \rangle, \langle \phi, \hat{\phi} \rangle^\perp)$ of $\langle \phi, \hat{\phi} \rangle$ by

$$\mathrm{d}\psi = \nabla^{\phi, \hat{\phi}} \psi + \beta \psi,$$

for $\psi \in \Gamma \langle \phi, \hat{\phi} \rangle$, and deduce the following Gauss equation from the flatness of d:

$$R^{\nabla^{\phi, \hat{\phi}}} = \beta^T \wedge \beta.$$

Thus flatness of $\langle \phi, \hat{\phi} \rangle$ amounts to the vanishing of $(\beta \hat{\phi} \wedge \beta \phi)$. However, (4.2) gives

$$\mathrm{d}\hat{\phi} - \alpha \hat{\phi} = \mathrm{d}\phi \circ r - \alpha \phi$$

with both sides palpably orthogonal to $\langle \phi, \hat{\phi} \rangle$ whence

$$\beta \hat{\phi} = \mathrm{d}\hat{\phi} - \alpha \hat{\phi}, \qquad \beta \phi = \mathrm{d}\phi - (\alpha \circ r^{-1})\phi$$

and, in particular, $\beta \hat{\phi} = (\beta \circ r)\phi$. Thus

$$(\beta \hat{\phi} \wedge \beta \phi) = ((\beta \circ r)\phi \wedge \beta \phi) = (\mathrm{d}\mathfrak{f} \circ r \wedge \mathrm{d}\mathfrak{f})$$

and we conclude:

Theorem 5.3. $\mathcal{N}_{f, \hat{f}}$ *is flat if and only if r is symmetric.*

That is $(f, \hat{f})$ is a Ribaucour pair of Legendre maps if and only if $(\mathfrak{f}, \hat{\mathfrak{f}})$ are a Ribaucour pair of submanifolds.

Suppose now that we are in the situation of the permutability theorem: thus we are given two Ribaucour transforms $\hat{\mathfrak{f}}_0, \hat{\mathfrak{f}}_1 : \Sigma^k \to S^{m+1}$ of $\mathfrak{f}_0 : \Sigma^k \to S^{m+1}$. We therefore have $r_0, r_1 \in \Gamma\mathrm{End}(T\Sigma^k)$ symmetric with respect to the metric induced by $\mathfrak{f}_0$.

Assume, once and for all, that these three maps are pairwise pointwise distinct. This ensures that $\hat{f}_0 \cap \hat{f}_1 = \{0\}$ so that we can define $V = \hat{f}_0 \oplus \hat{f}_1$.

Theorem 5.4. V *is flat if and only if $[r_0, r_1] = 0$.*

Proof. With $s_i = \hat{f}_i \cap f_0$ and $\hat{\phi}_i = \hat{\mathfrak{f}}_i + t_0$, $\hat{\phi}_i \circ \pi$ represent non-zero sections of $\hat{f}_i/s_i$ so that, by Lemma 3.5, V is flat if and only if $\big(R^\nabla(\hat{\phi}_0 \circ \pi), \hat{\phi}_1 \circ \pi \big)$ vanishes. Let $\beta_V \in \Omega_{M^m} \otimes \mathrm{Hom}(V, V^\perp)$ be the second fundamental form of V so that

$$\mathrm{d}\psi = \nabla \psi + \beta_V \psi,$$

for $\psi \in \Gamma V$. As before, a Gauss equation gives $R^\nabla = \beta_V^T \wedge \beta_V$ so that flatness of V amounts to the vanishing of $\big(\beta_V(\hat{\phi}_0 \circ \pi) \wedge \beta_V(\hat{\phi}_1 \circ \pi) \big)$.

On the other hand, we also have

$$\mathrm{d}\hat{\phi}_i = \nabla^{\phi_0, \hat{\phi}_i} \hat{\phi}_i + \beta^i \hat{\phi}_i$$

with $\beta^i \in \Omega^1_{\Sigma^k} \otimes \mathrm{Hom}(\langle \phi_0, \hat{\phi}_i \rangle, \langle \phi_0, \hat{\phi}_i \rangle^\perp)$. Now $\pi^{-1}\langle \phi_0, \hat{\phi}_i \rangle \subset V$ so that $(\pi^* \beta^i - \beta_V)\hat{\phi}_i \circ \pi$ takes values in V. Moreover, each $\hat{f}_i$ is Legendre so that $\mathrm{d}(\hat{\phi}_i \circ \pi) \perp s_i$ as is $\nabla^{\phi_0, \hat{\phi}_i} \hat{\phi}_i$. We conclude that $(\pi^* \beta^i - \beta_V)\hat{\phi}_i \circ \pi \perp \phi_0, \hat{\phi}_i, s_i$ and so takes values in $V \cap \langle \phi_0, \hat{\phi}_i, s_i \rangle^\perp = s_i$. Since $V^\perp, s_0, s_1$ are mutually orthogonal, this gives

$$\begin{aligned}
\big(\beta_V(\hat{\phi}_0 \circ \pi) \wedge \beta_V(\hat{\phi}_1 \circ \pi) \big) &= \pi^* \big(\beta^0 \hat{\phi}_0 \wedge \beta^1 \hat{\phi}_1 \big) \\
&= \pi^* \big((\beta^0 \circ r_0)\phi_0 \wedge (\beta^1 \circ r_1)\phi_0 \big) \\
&= \pi^* \big(\mathrm{d}\mathfrak{f}_0 \circ r_0 \wedge \mathrm{d}\mathfrak{f}_0 \circ r_1 \big)
\end{aligned}$$

whence the conclusion. $\qquad \square$

Let us suppose then that $[r_0, r_1]$ vanishes so that, by Theorems 3.4 and 5.4, the Permutability Theorem holds. It remains to show that all the (locally defined) Legendre maps $f_\alpha, \hat{f}_\beta : M^m \to \mathcal{Z}$ of Theorem 3.4 arise from maps $\mathfrak{f}_\alpha, \hat{\mathfrak{f}}_\beta : \Sigma^k \to S^{m+1}$. That is, each *point map* $f_\alpha \cap \langle t_1 \rangle^\perp, \hat{f}_\beta \cap \langle t_1 \rangle^\perp$ is constant on the fibres of π. One approach to this, valid on simply-connected subsets of M^m which have connected intersections with the fibres of π, is to compute derivatives of these point maps along said fibres. However, we employ an alternative, slightly indirect, argument which provides us with rather more information: we will show that all point maps in a Demoulin family arise as pull-backs of sections of a certain bundle of $(2, 1)$-planes in $\langle t_1 \rangle^\perp$ which are parallel with respect to a certain metric connection. The bundle is the same for each Demoulin family but the connections are different.

For all this, we begin by recalling the well-known fact that a 2-plane $\langle \sigma, \tau \rangle \subset V$ is null if and only if the 2-vector $\sigma \wedge \tau \in \wedge^2 V$ is self-dual or anti-self-dual. Choose the orientation on V for which $\wedge^2 f_0$ is self-dual and let $* : \wedge^2 V \to \wedge^2 V$ be the corresponding Hodge star operator with orthogonal eigenspace decomposition

$$\wedge^2 V = \wedge^2_+ V \oplus \wedge^2_- V.$$

The $(2, 2)$-metric on V induces a $(2, 4)$-metric from which both $\wedge^2_\pm V$ inherit a $(1, 2)$-metric. The flat metric connection ∇ on V induces flat metric connections $\nabla^\pm$ on $\wedge^2_\pm V$ and the isotropic parallel line subbundles with respect to ∇^+, respectively ∇^-, are the $\wedge^2 f_\alpha$, respectively $\wedge^2 \hat{f}_\beta$.

Now contemplate the bundle $U = \langle \phi_0, \hat{\phi}_0, \hat{\phi}_1 \rangle$: a bundle of $(2, 1)$-planes in $\langle t_1 \rangle^\perp$ over Σ^k. Note that $\pi^{-1} U = V \cap \langle t_1 \rangle^\perp$. Here is the geometry of U: under the diffeomorphism $x \mapsto \langle x + t_0 \rangle$ of S^{m+1} with $\mathcal{Q}_0$, $U(p) \cap \mathcal{Q}_0$ represents the intersection of S^{m+1} with the affine 2-plane containing the points $\mathfrak{f}_0(p), \hat{\mathfrak{f}}_0(p), \hat{\mathfrak{f}}_1(p)$, that is, the circle containing these points.

Choose $t \in \Gamma V$ with $(t, t) = -1$, $t \perp \pi^{-1} U$ so that $V = \pi^{-1} U \oplus \langle t \rangle$. For $\eta \in \wedge^2 V$, we note that the interior product $\iota_t \eta \perp t$ whence $\iota_t \eta \in \pi^{-1} U$. We therefore define $T_\pm : \wedge^2_\pm V \to \pi^{-1} U$ by

$$T_\pm \eta = \sqrt{2} \iota_t \eta.$$

Since $*$ is an involutive isometry that permutes $\wedge^2 \pi^{-1} U$ with $\pi^{-1} U \wedge \langle t \rangle$, we have:

Lemma 5.5. $T_\pm$ *is an anti-isometric[8] isomorphism with inverse*

$$T_\pm^{-1} \phi = \big(\phi \wedge t \pm *(\phi \wedge t) \big) / \sqrt{2}.$$

Moreover,

$$T_+ \wedge^2 f_\alpha = f_\alpha \cap \langle t_1 \rangle^\perp \qquad\qquad T_- \wedge^2 \hat{f}_\beta = \hat{f}_\beta \cap \langle t_1 \rangle^\perp.$$

We use $T_\pm$ to induce flat metric connections, also called $\nabla^\pm$, on $\pi^{-1} U$. By Lemma 5.5, the parallel isotropic line subbundles of $\pi^{-1} U$ with respect to ∇^+, respectively ∇^-, are the point maps $f_\alpha \cap \langle t_1 \rangle^\perp$, respectively $\hat{f}_\beta \cap \langle t_1 \rangle^\perp$. This means we will be done as soon as we prove

Proposition 5.6. *There are flat metric connections $D^\pm$ on U such that*

$$\nabla^\pm = \pi^{-1} D^\pm.$$

For this, we need the following lemma which is surely well-known (and, in any case, a straight-forward exercise to prove):

---

[8]Thus $(T_\pm \eta_1, T_\pm \eta_2) = -(\eta_1, \eta_2)$.

Lemma 5.7. *Let* $\pi : M^m \to \Sigma^k$ *be a bundle with connected fibres,* $U \to \Sigma^k$ *a vector bundle and* ∇ *a connection on* $\pi^{-1}U$. *Then* $\nabla = \pi^{-1}D$ *for some connection* D *on* U *if and only if, for all* $X \in \ker \mathrm{d}\pi$ *and* $\phi \in \Gamma U$,

(i) $\nabla_X(\phi \circ \pi) = 0$;
(ii) $\iota_X R^\nabla = 0$.

With this in hand, we compute: for $\psi \in \Gamma\pi^{-1}U$,

$$
\begin{aligned}
\nabla^\pm\psi &= \iota_t \nabla\big(\psi \wedge t \pm *(\psi \wedge t)\big) \\
(5.3) \qquad &= \iota_t\big(\nabla\psi \wedge t + \psi \wedge \nabla t \pm *(\nabla\psi \wedge t + \psi \wedge \nabla t)\big) \\
&= \pi^{-1}\nabla^U\psi \pm \iota_t * (\psi \wedge \nabla t)
\end{aligned}
$$

where ∇^U is the connection on U induced by d (whence $\pi^{-1}\nabla^U\psi$ is the U-component of $\nabla\psi$). Now let $X \in \ker \mathrm{d}\pi$ and $\phi \in \Gamma U$. We know that $\nabla_X(\phi \circ \pi)$ is the V-component of $\mathrm{d}_X(\phi\circ\pi)$ and so vanishes. It follows that ∇_X preserves $\pi^{-1}U$ whence $\nabla_X t = 0$. Since both $\nabla^\pm$ are flat, Lemma 5.7 assures us that Proposition 5.6 holds.

We have therefore arrived at the following situation: on a simply connected open subset $\Omega \subset \Sigma^k$, we have isotropic line subbundles $\langle\phi_\alpha\rangle$, parallel for D^+, and $\langle\hat\phi_\beta\rangle$, parallel for D^-, so that

$$
f_\alpha \cap \langle t_1\rangle^\perp = \langle\phi_\alpha \circ \pi\rangle \qquad\qquad \hat f_\beta \cap \langle t_1\rangle^\perp = \langle\hat\phi_\beta \circ \pi\rangle.
$$

Define $\mathfrak{f}_\alpha, \hat{\mathfrak{f}}_\beta : \Omega \to S^{m+1}$ by

$$
\langle\mathfrak{f}_\alpha + t_0\rangle = \langle\phi_\alpha\rangle \qquad\qquad \langle\hat{\mathfrak{f}}_\beta + t_0\rangle = \langle\hat\phi_\beta\rangle
$$

and finally deduce the Bianchi Permutability Theorem for Ribaucour transforms of maps $\Sigma^k \to S^{m+1}$:

Theorem 5.8. *Let* $\mathfrak{f}_0 : \Sigma^k \to S^{m+1}$ *be an immersion of a simply connected manifold with* $\hat{\mathfrak{f}}_0, \hat{\mathfrak{f}}_1$ *pointwise distinct Ribaucour transforms satisfying* $[r_0, r_1] = 0$. *Then, for* $\alpha, \beta \in \mathbb{RP}^1$, *there are maps* $\mathfrak{f}_\alpha, \hat{\mathfrak{f}}_\beta : \Sigma^k \to S^{m+1}$ *such that:*

(i) $\mathfrak{f}_0 \in \{\mathfrak{f}_\alpha\}$; $\hat{\mathfrak{f}}_0, \hat{\mathfrak{f}}_1 \in \{\hat{\mathfrak{f}}_\beta\}$;
(ii) *each* $\mathfrak{f}_\alpha$ *is a Ribaucour transform of each* $\hat{\mathfrak{f}}_\beta$;
(iii) *for each* $p \in \Sigma^k$, *the points* $\mathfrak{f}_\alpha(p), \hat{\mathfrak{f}}_\beta(p)$, $\alpha, \beta \in \mathbb{RP}^1$, *are concircular;*
(iv) *any four maps in one Demoulin family, either* $\{\mathfrak{f}_\alpha\}$ *or* $\{\hat{\mathfrak{f}}_\beta\}$, *have constant cross-ratio.*

Proof. Only the last point requires further elaboration: any $\mathfrak{f}_\alpha + t_0$ spans a D^+-parallel line bundle which admits a parallel section. The cross-ratio of four such maps can be computed in terms of the inner products between these parallel sections [21, §6.5.4] and so is constant since D^+ is metric. $\square$

Remarks.

1. Assertion (iii), that corresponding points of the maps of a Bianchi quadrilateral lie on circles, provides a more direct approach to Theorem 3.7 in the context of Möbius geometry: one can construct the eighth map in a Bianchi cube via Miguel's theorem. This gives a link between the Bianchi Permutability Theorem and the theory of "discrete orthogonal nets", or "circular nets", see for example [5].
2. That corresponding points of members of a single Demoulin family are concircular is due to Demoulin [15] while Bianchi showed that the circles

for the two families coincide [3, §354]. Assertion (iv) on cross-ratios is also due to Demoulin [16].

Remark. Observe that (5.3) tells us that $D^{\pm}$ are of the form

$$D^{\pm} = \nabla^U \pm B$$

for some $B \in \Omega^1 \otimes \mathfrak{o}(U)$. Moreover, by construction, $\langle \phi_0 \rangle$ is D^+-parallel while $\langle \hat{\phi}_0 \rangle, \langle \hat{\phi}_1 \rangle$ are D^--parallel. These properties fix B (and so $D^{\pm}$) uniquely: the difference of two such B would be $\mathfrak{o}(U)$-valued while preserving the decomposition $\langle \hat{\phi}_0 \rangle \oplus \langle \phi_0 \rangle \oplus \langle \hat{\phi}_1 \rangle$ and so must vanish. This suggests an alternative approach to the Permutability Theorem entirely in the context of conformal geometry

6. Example

We conclude by presenting a very simple configuration of two Ribaucour transforms $\hat{\mathfrak{f}}_0$ and $\hat{\mathfrak{f}}_1$ of a surface $\mathfrak{f}_0$ in S^3, where the Bianchi Permutability Theorem can fail. For this, the curvature directions of $\mathfrak{f}_0$ should be ambiguous while those of $\hat{\mathfrak{f}}_0$ and $\hat{\mathfrak{f}}_1$ should be well-defined and different. Thus $\mathfrak{f}_0$ should parametrise (part of) a 2-sphere s and we will take the $\hat{\mathfrak{f}}_i$ to be Dupin cyclides as these are the simplest surfaces in Lie sphere geometry. Recall that all Dupin cyclides are equivalent in Lie sphere geometry: they are congruent to a circle (or, equivalently, a torus of revolution) [24].

Thus, let $\mathcal{Q}$ be the projective light-cone of $\mathbb{R}^{4,2}$ and fix a unit time-like $t_1 \in \mathbb{R}^{4,2}$ to get a space

$$\mathcal{Q}_0 = \mathcal{Q} \cap t_1^{\perp}$$

of point spheres. We can then write $s = \langle e + t_1 \rangle$ with e a space-like unit vector in $\langle t_1 \rangle^{\perp}$.

6.1. The first Ribaucour transform. We fix two points on s:

$$p_0, p_\infty \perp t_1, s, \quad |p_0|^2 = |p_\infty|^2 = 0, \quad (p_0, p_\infty) = -1,$$

and choose an orthonormal basis (e_1, e_2) for $\langle t_1, e, p_0, p_\infty \rangle^{\perp}$. Geometrically, $t_1 + e_1$ and $t_1 + e_2$ define two 2-spheres that contain the points $\langle p_0 \rangle$ and $\langle p_\infty \rangle$. The circle in which these spheres intersect is a (degenerate) Dupin cyclide which we take as our first Ribaucour transform. Thus we define a Legendre map

$$\hat{f}_0 = \langle \hat{\kappa}_{01}, \hat{\kappa}_{02} \rangle$$

where

$$(u, v) \mapsto \hat{\kappa}_{01}(v) := t_1 + \cos v \, e_1 + \sin v \, e_2$$

$$(u, v) \mapsto \hat{\kappa}_{02}(u) := p_0 + u \, e + \frac{u^2}{2} \, p_\infty \perp t_1.$$

A section of the corresponding point map is given by $\hat{\phi}_0 := \hat{\kappa}_{02}$.

We parametrise (the contact lift of) s by $f_0 := \langle s, \phi_0 \rangle$, where (a section of) the point map is given by

$$\phi_0(u, v) := p_0 + u \, (\cos v \, e_1 + \sin v \, e_2) + \frac{u^2}{2} \, p_\infty \perp t_1.$$

Then the section

$$\sigma_0 = (\hat{\kappa}_{01}, s) \hat{\kappa}_{02} - (\hat{\kappa}_{02}, s) \hat{\kappa}_{01} \in \Gamma(\hat{f}_0)$$

defines a sphere congruence that is enveloped by both $\hat{f}_0$ and f_0 since $\sigma_0(u, v) = u \, s + \phi_0(u, v)$.

Now define

$$r_0 := \frac{\partial}{\partial u} \, \mathrm{d}u \quad \text{and} \quad \alpha := \frac{1}{u} \, \mathrm{d}u$$

and note that

$$\mathrm{d}\hat{\phi}_0 - \alpha \, \hat{\phi}_0 = \frac{1}{u}\left(-p_0 + \frac{u^2}{2} p_\infty\right) \mathrm{d}u = \mathrm{d}\phi_0 \circ r_0 - \alpha \, \phi_0.$$

We see that r_0 is symmetric with respect to the metric $\mathrm{d}u^2 + u^2 \mathrm{d}v^2$ induced by ϕ_0 and so conclude that $\hat{f}_0$ is a Ribaucour transform[9] of f_0.

6.2. The second Ribaucour transform.
Now fix two possibly different points on s:

$$\tilde{p}_0, \tilde{p}_\infty \perp t_1, s, \quad |\tilde{p}_0|^2 = |\tilde{p}_\infty|^2 = 0, \quad (\tilde{p}_0, \tilde{p}_\infty) = -1,$$

and choose an orthonormal basis $(\tilde{e}_1, \tilde{e}_2)$ for $\langle t_1, e, \tilde{p}_0, \tilde{p}_\infty \rangle^\perp$. Then $s + \tilde{p}_0$ and $s + \tilde{p}_\infty$ define two 2-spheres which touch s at the points $\langle \tilde{p}_0 \rangle$ and $\langle \tilde{p}_\infty \rangle$ respectively. These spheres intersect in a circle which we take as our second Ribaucour transform.

Thus we define a Legendre map by

$$\hat{f}_1 = \langle \hat{\kappa}_{11}, \hat{\kappa}_{12} \rangle$$

where

$$(u, v) \mapsto \hat{\kappa}_{11}(v) := (e + \tilde{p}_0 + \tilde{p}_\infty) + \cos v \, \tilde{e}_1 + \sin v \, \tilde{e}_2 \perp t_1$$

$$(u, v) \mapsto \hat{\kappa}_{12}(u) := (1 - u + \frac{u^2}{2}) t_1 + (1 - u)(e + \tilde{p}_0) + (-u + \frac{u^2}{2})(e + \tilde{p}_\infty).$$

Then $\hat{\phi}_1 := \hat{\kappa}_{11}$ is a section of the corresponding point map.

Now we parametrise s by $\tilde{f}_0 := \langle s, \tilde{\phi}_0 \rangle$ with point map

$$\tilde{\phi}_0(u, v) := \tilde{p}_0 + u \, (\cos v \, \tilde{e}_1 + \sin v \, \tilde{e}_2) + \frac{u^2}{2} \tilde{p}_\infty \perp t_1.$$

Then the section

$$\sigma_1 = (\hat{\kappa}_{11}, s)\hat{\kappa}_{12} - (\hat{\kappa}_{12}, s)\hat{\kappa}_{11} \in \Gamma(\hat{f}_1)$$

defines a sphere congruence that is enveloped by $\hat{f}_1$ and $\tilde{f}_0$ since $\sigma_1(u, v) = (1 - u + \frac{u^2}{2}) s + \tilde{\phi}_0(u, v)$.

Now define

$$r_1 := \frac{1}{u} \frac{\partial}{\partial v} \, \mathrm{d}v \quad \text{and} \quad \alpha := 0$$

and note that

$$\mathrm{d}\hat{\phi}_1 - \alpha \, \hat{\phi}_1 = (-\sin v \, \tilde{e}_1 + \cos v \, \tilde{e}_2) \, \mathrm{d}v = \mathrm{d}\tilde{\phi}_0 \circ r_1 - \alpha \, \tilde{\phi}_0.$$

Again, r_1 is symmetric with respect to the metric $\mathrm{d}u^2 + u^2 \mathrm{d}v^2$ induced by $\tilde{\phi}_0$ showing that $\hat{f}_0$ is a Ribaucour transform of $\tilde{f}_0$.

---

[9]Alternatively, $e + t_1$ represents a parallel section of $\mathcal{N}_{f_0 \hat{f}_0}$.

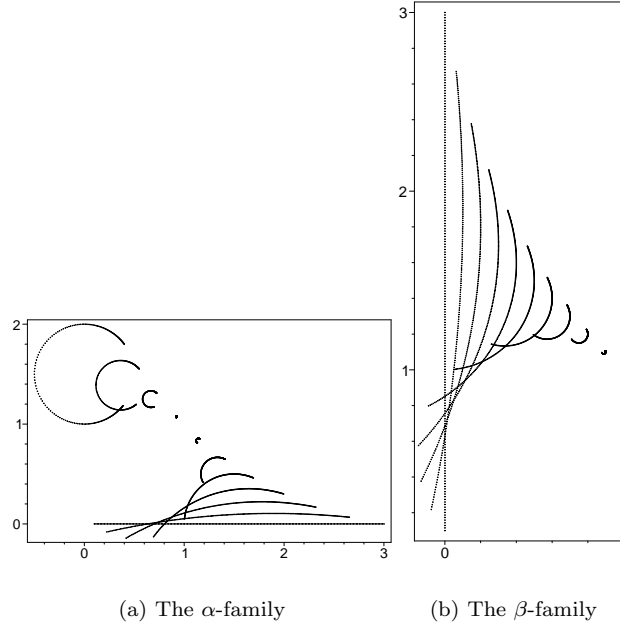(a) The α-family   (b) The β-family

FIGURE 3. Demoulin families of Dupin cyclides

6.3. **The Permutability Theorem.** Now let us see when the Bianchi Permutabil-
ity Theorem holds. For this we should choose a common parametrisation of all par-
ticipating surfaces but we can avoid this issue by noting that, for both Ribaucour
transforms, the images of the eigendirections of r_0 and r_1 under the parametrisa-
tions $\phi_0, \tilde{\phi}_0$ of s are tangent to systems of circles on s passing through p_0, p_∞ or
$\tilde{p}_0, \tilde{p}_\infty$, respectively, together with their orthogonal circles. The two endomorphisms
can only have the same eigendirections (and so commute) if these two circle systems
coincide, that is, if

$$\{\tilde{p}_0, \tilde{p}_\infty\} = \{p_0, p_\infty\}.$$

Thus, in the generic case where this condition is not met, the Bianchi Permutability
Theorem fails.

On the other hand, if we choose $\tilde{p}_0 = p_0$ and $\tilde{p}_\infty = p_\infty$ then $\tilde{\phi}_0 = \phi_0$, $[r_0, r_1] = 0$,
and the Bianchi Permutability Theorem holds. To exhibit the Demoulin families,
it only remains to determine four parallel sections of $V = \hat{f}_0 \oplus \hat{f}_1$. In the case at
hand, this can be done by inspection [10] and it is then a matter of linear algebra
(which can be delegated to a computer algebra engine) to determine the Demoulin
families.

It turns out that both families consist of Dupin cyclides apart from two spheres s
and $\tilde{s} = \langle t_1 - 3e - 2(p_0 + p_\infty) \rangle$ in the α-family (containing f_0) and one sphere (also $\tilde{s}$
but parametrised differently) in the β-family (containing $\hat{f}_0$ and $\hat{f}_1$). After suitable
stereographic projection s becomes a plane, say $z = 0$, $\hat{f}_0$ becomes a vertical line,

---

[10] $\hat{\sigma}_0 := (1 - 2u + \frac{u^2}{2})\hat{\kappa}_{01} + \frac{1}{2}\sigma_0$ and $\hat{\sigma}_1 := (1 - 2u + \frac{u^2}{2})\hat{\kappa}_{12} + \frac{1}{2}\sigma_1$ complement the sections
σ_0 and σ_1 to give sections of V which are parallel up to a common scaling by $1/(1 - 2u + \frac{u^2}{2})$, as
one easily verifies.

say $x = y = 0$, and all Dupin cyclides become surfaces of revolution with that line as axis[11]: their meridian curves in the $y = 0$-plane are shown in Figure 3.

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[11]In particular, $\hat{f}_1$ is a circle parallel to f_0.

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